

REGULAR ORBITS OF FINITE PRIMITIVE SOLVABLE GROUPS, III

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ABSTRACT. Suppose that a finite solvable group G acts faithfully, irreducibly and quasi-primitively on a finite vector space V . Then G has a uniquely determined normal subgroup E which is a direct product of extraspecial p -groups for various p and we denote $e = \sqrt{|E/\mathbf{Z}(E)|}$. We prove that when $e = 2, 3, 4, 8, 9, 16$, G will have regular orbits on V when the corresponding vector space is not too small.

1. INTRODUCTION

Let G be a finite group and V a finite, faithful and completely reducible G -module. It is a classical theme to study orbit structures of G acting on V . One of the most important and natural questions about orbit structure is to establish the existence of an orbit of a certain size. For a long time, there has been a deep interest and need to examine the size of the largest possible orbits in linear group actions. The orbit $\{v^g \mid g \in G\}$ is called regular, if $\mathbf{C}_G(v) = 1$ holds or equivalently the size of the orbit v^G is $|G|$.

In [3], Pálffy and Pyber asked if it is possible to classify all pairs A, G with $(|A|, |G|) = 1$ such that $A \leq \text{Aut}(G)$ has a regular orbit on G . While the task is pretty challenge, at least for primitive solvable linear groups, we can say something along this line.

Suppose that a finite solvable group G acts faithfully, irreducibly and quasi-primitively on a finite vector space V . Then G has a uniquely determined normal subgroup E which is a direct product of extraspecial p -groups for various p . We denote $e = \sqrt{|E/\mathbf{Z}(E)|}$ (an invariant measuring the complexity of the group). It is proved in [6, Theorem 3.1] and [7, Theorem 3.1] that if $e = 5, 6, 7$ or $e \geq 10$ and $e \neq 16$, then G will have regular orbits on V . Also, when $e = 1, 2, 3, 4, 8, 9$ or 16 , there are examples where G has no regular orbit on V .

As suggested by the referee of the previous paper [7], it would be interesting and useful to give more detailed analysis for all those smaller e values. In this paper, we attempt to push the limit of our calculations. As one may expect, when the e are small, the calculation gets more tedious and complicated. The bound we obtain here is not perfect, but hopefully is good enough for most applications. I would hope that in the near future, by refining the arguments or relying on the developments of the computer program [1], one might be able to obtain a complete classifications up to group isomorphisms and representation isomorphisms.

Due to the nature of this question, when e are in those exceptional small cases, the example where G does not have a regular orbit only happens when the sizes of the vector spaces are relative small. In this paper, we study the remaining small cases, and show that for $e = 2, 3, 4, 8, 9, 16$, G will have a regular orbit on V when the corresponding $|V|$ is not too small.

2. NOTATION AND LEMMAS

Notation:

- (1) Let G be a finite group, let S be a subset of G and let π be a set of different primes. For each prime p , we denote $\text{SP}_p(S) = \{\langle x \rangle \mid o(x) = p, x \in S\}$ and $\text{EP}_p(S) = \{x \mid o(x) = p, x \in S\}$. We denote $\text{SP}(S) = \bigcup \text{SP}_p(S)$, $\text{EP}(S) = \bigcup \text{EP}_p(S)$ and $\text{EP}_\pi(S) = \bigcup_{p \in \pi} \text{EP}_p(S)$. We denote $\text{NEP}(S) = |\text{EP}(S)|$, $\text{NEP}_p(S) = |\text{EP}_p(S)|$ and $\text{NEP}_\pi(S) = |\text{EP}_\pi(S)|$.
- (2) Let n be an even integer, q a power of a prime. Let V be a standard symplectic vector space of dimension n of \mathbb{F}_q . We use $\text{SCRSp}(n, q)$ or $\text{SCRSp}(V)$ to denote the set of all solvable subgroups of $\text{Sp}(V)$ which acts completely reducibly on V . We use $\text{SIRSp}(n, q)$ or $\text{SIRSp}(V)$ to denote the set of all solvable subgroups of $\text{Sp}(V)$ which acts irreducibly on V . Define $\text{SCRSp}(n_1, q_1) \times \text{SCRSp}(n_2, q_2) = \{H \times I \mid H \in \text{SCRSp}(n_1, q_1) \text{ and } I \in \text{SCRSp}(n_2, q_2)\}$.
- (3) Let V be a finite vector space and let $G \subseteq \text{GL}(V)$. We define $\text{PC}(G, V, p, i) = \{x \mid x \in \text{EP}_p(G) \text{ and } \dim(\mathbf{C}_V(x)) = i\}$ and $\text{NPC}(G, V, p, i) = |\text{PC}(G, V, p, i)|$. We will drop V in the notation when it is clear in the context.
- (4) If V is a finite vector space of dimension n over $\text{GF}(q)$, where q is a prime power, we denote by $\Gamma(q^n) = \Gamma(V)$ the semi-linear group of V , i.e.,

$$\Gamma(q^n) = \{x \mapsto ax^\sigma \mid x \in \text{GF}(q^n), a \in \text{GF}(q^n)^\times, \sigma \in \text{Gal}(\text{GF}(q^n)/\text{GF}(q))\},$$

and we define

$$\Gamma_0(q^n) = \{x \mapsto ax \mid x \in \text{GF}(q^n), a \in \text{GF}(q^n)^\times\}.$$

- (5) We use $H \wr S$ to denote the wreath product of H with S where H is a group and S is a permutation group.
- (6) We use $\mathbf{F}(G)$ to denote the Fitting subgroup of G .
- (7) Let n be a positive integer, we use $\text{Div}(n)$ to denote the number of different prime divisors of n . Clearly if $n = 1$, $\text{Div}(n) = 0$.
- (8) Let n be a positive integer and p be a prime, we set $D_p(n) = 0$ if $p \nmid n$ and $D_p(n) = p$ if $p \mid n$.

Definition 2.1. Suppose that a finite solvable group G acts faithfully, irreducibly and quasi-primitively on a finite vector space V . Let $\mathbf{F}(G)$ be the Fitting subgroup of G and $\mathbf{F}(G) = \prod_i P_i$, $i = 1, \dots, m$ where P_i are normal p_i -subgroups of G for different primes p_i . Let $Z_i = \Omega_1(\mathbf{Z}(P_i))$. We define

$$E_i = \begin{cases} \Omega_1(P_i) & \text{if } p_i \text{ is odd;} \\ [P_i, G, \dots, G] & \text{if } p_i = 2 \text{ and } [P_i, G, \dots, G] \neq 1; \\ Z_i & \text{otherwise.} \end{cases}$$

By proper reordering we may assume that $E_i \neq Z_i$ for $i = 1, \dots, s$, $0 \leq s \leq m$ and $E_i = Z_i$ for $i = s + 1, \dots, m$. We define $E = \prod_{i=1}^s E_i$, $Z = \prod_{i=1}^s Z_i$ and we define $\bar{E}_i = E_i/Z_i$, $\bar{E} = E/Z$. Furthermore, we define $e_i = \sqrt{|E_i/Z_i|}$ for $i = 1, \dots, s$ and $e = \sqrt{|E/Z|}$.

Theorem 2.2. Suppose that a finite solvable group G acts faithfully, irreducibly and quasi-primitively on an n -dimensional finite vector space V over finite field \mathbb{F} of characteristic r . We use the notation in Definition 2.1. Then every normal abelian subgroup of G is cyclic and G has normal subgroups $Z \leq U \leq F \leq A \leq G$ such that,

- (1) $F = EU$ is a central product where $Z = E \cap U = \mathbf{Z}(E)$ and $\mathbf{C}_G(F) \leq F$;
- (2) $F/U \cong E/Z$ is a direct sum of completely reducible G/F -modules;

- (3) E_i is an extraspecial p_i -group for $i = 1, \dots, s$ and $e_i = p_i^{n_i}$ for some $n_i \geq 1$. Furthermore $(e_i, e_j) = 1$ when $i \neq j$ and $e = e_1 \dots e_s$ divides n , also $\gcd(r, e) = 1$;
- (4) $A = \mathbf{C}_G(U)$ and $G/A \lesssim \text{Aut}(U)$, A/F acts faithfully on E/Z ;
- (5) $A/\mathbf{C}_A(E_i/Z_i) \lesssim \text{Sp}(2n_i, p_i)$;
- (6) U is cyclic and acts fixed point freely on W where W is an irreducible submodule of V_U ;
- (7) $|V| = |W|^{eb}$ for some integer b ;
- (8) G/A is cyclic and $|G : A| \mid \dim(W)$. $G = A$ when $e = n$;
- (9) Let $g \in G \setminus A$, assume that $o(g) = t$ where t is a prime and let $|W| = r^m$. Then $t \mid m$ and we can view the action of g on U as follows, $U \subseteq \mathbb{F}_{r^m}^*$ and $g \in \text{Gal}(\mathbb{F}_{r^m} : \mathbb{F}_r)$.

Proof. This is [7, Theorem 2.2]. □

Lemma 2.3. *Suppose that a finite solvable group G acts faithfully, irreducibly and quasi-primitively on a finite vector space V . Using the notation in Theorem 2.2, we have $|G| \leq \dim(W) \cdot |A/F| \cdot e^2 \cdot (|W| - 1)$.*

Proof. By Theorem 2.2, $|G| = |G/A||A/F||F|$ and $|F| = |E/Z||U|$. Since $|G/A| \mid \dim(W)$, $|E/Z| = e^2$ and $|U| \mid (|W| - 1)$, we have $|G| \mid \dim(W) \cdot |A/F| \cdot e^2 \cdot (|W| - 1)$. □

Lemma 2.4. *Suppose that a finite solvable group G acts faithfully and quasi-primitively on a finite vector space V over the field \mathbb{F} . Let $g \in \text{EP}_s(G)$ and we use the notation in Theorem 2.2.*

- (1) *If $g \in F$ then $|\mathbf{C}_V(g)| \leq |W|^{\frac{1}{s}eb}$.*
- (2) *If $g \in A \setminus F$ then $|\mathbf{C}_V(g)| \leq |W|^{\lfloor \frac{3}{4}e \rfloor b}$.*
- (3) *If $g \in A \setminus F$, $s \geq 3$ and $s \nmid |E|$, then $|\mathbf{C}_V(g)| \leq |W|^{\lfloor \frac{1}{2}e \rfloor b}$.*
- (4) *If $g \in A \setminus F$, $s = 2$ and $s \nmid |E|$, then $|\mathbf{C}_V(g)| \leq |W|^{\lfloor \frac{2}{3}e \rfloor b}$.*
- (5) *If $g \in G \setminus A$ then $|\mathbf{C}_V(g)| \leq |W|^{\frac{1}{s}eb}$.*

Proof. (1) is a slight improvement of [7, Lemma 2.4(1)]. We only need to consider the cases when $s \geq 3$. Assume first that $\langle g \rangle \leq F$, $\langle g \rangle \not\leq U$. Then $s \neq r$ and $\langle g \rangle \leq E \triangleleft G$, with E an extraspecial s -group. We consider the restriction V_E . It is a direct sum of faithful irreducible E -modules, since $\mathbf{Z}(E) \leq U$ acts fixed point freely on V . As dimensions of centralizers do not change by extending the ground field and they add up in direct sums, we can assume that V is an irreducible, faithful E -module on an algebraically closed field \mathbb{K} . If χ is the (Brauer) character corresponding to V , then, as $r = \text{char}(\mathbb{K}) \neq s$, $\dim_{\mathbb{K}}(\mathbf{C}_V(S)) = [\chi_S, 1_S] = \frac{1}{s} \dim_{\mathbb{K}}(V)$ since $\chi(x) = 0$ for every $x \in E \setminus Z(E)$, and (1) is proved.

(2), (3), (4) and (5) follow from [7, Lemma 2.4]. □

Lemma 2.5. *Let G be a finite solvable group and V be a finite, faithful irreducible $\mathbb{F}G$ -module with dimension $\prod p_i^{n_i}$ where p_i are different primes. \mathbb{F} is algebraically closed and $\text{char}(\mathbb{F}) = s$ where $(s, \prod p_i) = 1$, E is a direct product of normal extraspecial subgroups E_i 's of G and $|E_i| = p_i^{2n_i+1}$. Define $Z_i = \mathbf{Z}(E_i)$ and $Z = \prod Z_i$. Consider $x \in G$, x is of prime order different than the characteristic of V and x acts trivially on Z . In [2] Isaacs defined good element. Let $C/Z = \mathbf{C}_{E/Z}(x)$, in our situation, x is a good element if $[x, C] = 1$. We call an element bad if it is not good. We have the following:*

- (1) If x is a good element, then we have that the Brauer character of x on V , say $\chi(x)$, is such that $|\chi(x)|^2 = |\mathbf{C}_{E/Z}(x)|$.
- (2) If x is a bad element, then $\chi(x) = 0$.

Proof. By [2, Theorem 3.5]. □

Lemma 2.6. Suppose that a finite solvable group G acts faithfully, irreducibly and quasi-primitively on a finite vector space V over a field \mathbb{F} . Using the notation in Theorem 2.2, let $x \in \text{EP}_s(A \setminus F)$ and $(s, \text{char}\mathbb{F}) = 1$. Let $C/Z = \mathbf{C}_{E/Z}(x)$, we call x a good element if $[x, C] = 1$, we call x a bad element if it is not good.

- (1) Assume x is a bad element and let $\beta = e/s$, then $|\mathbf{C}_V(x)| \leq |W|^{\beta b}$.
- (2) Assume x is a good element and $|\mathbf{C}_{E/Z}(x)| \leq a$, let

$$\beta = \lfloor \frac{1}{s}(e + (s-1)a^{1/2}) \rfloor$$

then $|\mathbf{C}_V(x)| \leq |W|^{\beta b}$.

- (3) Assume $o(x) = 2$ and x is a good element, then $|\mathbf{C}_{E/Z}(x)|$ is the square of an integer. Assume now that $2 \mid e$, then $|\mathbf{C}_{E/Z}(x)|$ is the square of an even integer.

Proof. This is [6, Lemma 2.6]. □

Lemma 2.7. Assume G satisfies Theorem 2.2 and we adopt the notation in it. Let p be a prime and $x \in \text{EP}_p(A \setminus F)$ and assume $|\mathbf{C}_{E/Z}(x)| = \prod_i p_i^{m_i}$. Define $U_p = \gcd(|U|, p)$. We have the following,

- (1) $\text{NEP}_p(A \setminus F) \leq \text{NEP}_p(A/F)|F|$.
- (2) $\text{NEP}_p(A \setminus F) \leq \frac{\text{NEP}_p(A/F)|F|}{\prod_{p_i \neq p} p_i^{m_i}}$.
- (3) $\text{NEP}_p(xF) \leq \prod_i M_i \cdot U_p$ where

$$M_i = \begin{cases} p_i^{2n_i} & \text{if } p = p_i \neq 2; \\ p_i^{2n_i - m_i} & \text{if } p \neq p_i; \\ 2^{m_i} & \text{if } p = p_i = 2. \end{cases}$$

- (4) Assume that $p = 2$ and x is a good element. Define $S = \{y \mid y \in \text{EP}_2(xF) \text{ and } y \text{ is a good element}\}$, then $|S| \leq \prod_i M_i \cdot U_2$ where

$$M_i = \begin{cases} p_i^{2n_i - m_i} & \text{if } p_i \neq 2; \\ 2^{m_i} & \text{if } p_i = 2 \text{ and } n_i \geq m_i; \\ 2^{2n_i - m_i} & \text{if } p_i = 2 \text{ and } n_i < m_i. \end{cases}$$

Proof. (1) and (2) follow from [6, Lemma 2.7]. (3) and (4) are slight improvements of [6, Lemma 2.7]. We only show the proof of (3) here. By the proof of [6, Lemma 2.7(3)], we know that $\text{NEP}_p(xF/U) \leq M_i$. Let $\alpha \in A$ and $o(\alpha) = p$, we consider $\text{NEP}_p(\alpha U)$. Since $U \subseteq \mathbf{Z}(A)$, $\text{NEP}_p(\alpha U) \leq U_p$ and the result follows. (4) can be proved similarly. □

Lemma 2.8. Let G be a finite solvable group and E a normal extraspecial subgroup of G . Assume that $|E| = p^{2n+1}$, p is odd and define $Z = \mathbf{Z}(E)$, $\bar{E} = E/Z$. Suppose that $\alpha \in G \setminus E$, $o(\alpha) = 2$ and $\mathbf{C}_Z(\alpha) = 1$, then $\text{NEP}_2(\alpha E) \leq p^{n+1}$.

Proof. Set $H = \langle \alpha, E \rangle$. Then H is a subgroup of G . Since $\mathbf{C}_E(\alpha) \neq E$, we can view H as a semi-direct product of E with α and $\alpha \in \text{Aut}(E)$. Consider $\bar{E} = E/Z$, let $\bar{I} = \{\bar{e} \in \bar{E} \mid \alpha(\bar{e}) = \bar{e}^{-1}\}$ and let I be the preimage of \bar{I} . Any elements of E inverted by α is contained

in I . Since $\mathbf{C}_I(\alpha) = 1$, I is abelian and $|I| \leq p^{n+1}$. Assume $o(\alpha e) = 2$ for $e \in E$, then $e \in I$ and thus $\text{NEP}_2(\alpha E) \leq p^{n+1}$. \square

Lemma 2.9. *Let G be a finite solvable group and U a normal cyclic subgroup of G . Let $\alpha \in G \setminus U$ and $o(\alpha) = 2$. Assume we can view the action of α on U as follows, $U \subseteq \mathbb{F}_{q^{2n}}^*$ and $\alpha \in \text{Gal}(\mathbb{F}_{q^{2n}} : \mathbb{F}_q)$. Then $\text{NEP}_2(\alpha U) \leq q^n + 1$.*

Proof. Let $u \in U$ and assume $o(\alpha u) = 2$, then $\alpha u \alpha u = 1$, $u^{q^n} \cdot u = u^{q^n+1} = 1$. \square

Lemma 2.10. *Let G be a finite solvable group and U a normal cyclic subgroup of G . Let $\alpha \in G \setminus U$ and $o(\alpha) = s$. Assume we can view the action of α on U as follows, $U \subseteq \mathbb{F}_{q^{sn}}^*$ and $\alpha \in \text{Gal}(\mathbb{F}_{q^{sn}} : \mathbb{F}_q)$. Then $\text{NEP}_s(\alpha U) \leq \frac{q^{sn}-1}{q^n-1}$.*

Proof. Let $u \in U$ and assume $o(\alpha u) = s$, then $\alpha u \alpha u \dots \alpha u = 1$.

This implies that $\alpha u \alpha^{-1} \alpha^2 u \alpha^{-2} \dots u \alpha^{s-1} \alpha^s u = 1$ and $u^{q^n} \cdot u^{q^{2n}} \dots u^{q^{(s-1)n}} \cdot u = u^{\frac{q^{sn}-1}{q^n-1}} = 1$. \square

Lemma 2.11. *Let G be a finite group and $K \cong S_3$ a normal subgroup of G . Suppose that $\alpha \in G \setminus K$, $o(\alpha) = 2$ then $\text{NEP}_2(\alpha K) \leq 3$.*

Proof. Set $H = \langle \alpha, K \rangle$. Then H is a subgroup of G . If $\mathbf{C}_K(\alpha) = K$, then $H \cong K \times \langle \alpha \rangle$ and $\text{NEP}_2(\alpha K) = 3$. If $\mathbf{C}_K(\alpha) \neq K$, we can view H as a semi-direct product of K with α and $\alpha \in \text{Aut}(K)$. Since $\text{Aut}(S_3) \cong S_3$, this cannot happen. \square

Lemma 2.12. *Let G be a finite group and $K \cong \text{SL}(2, 3)$ a normal subgroup of G . Suppose that $\alpha \in G \setminus K$, $o(\alpha) = 2$ then $\text{NEP}_2(\alpha K) \leq 12$.*

Proof. Set $H = \langle \alpha, K \rangle$. Then H is a subgroup of G . If $\mathbf{C}_K(\alpha) = K$, then $H \cong K \times \langle \alpha \rangle$ and $\text{NEP}_2(\alpha K) = 2$. If $\mathbf{C}_K(\alpha) \neq K$, we can view H as a semi-direct product of K with α and $\alpha \in \text{Aut}(K)$. Since $\text{Aut}(\text{SL}(2, 3)) \cong S_4 \cong \text{GL}(2, 3)/\mathbf{Z}(\text{GL}(2, 3))$, $H \cong \text{GL}(2, 3)$ and the result is clear. \square

Lemma 2.13. *Assume G satisfies Theorem 2.2 and we adopt the notation in it. We provide upper bound for the $\text{NEP}(G \setminus A)$. Since G/A is cyclic, we know that $\text{NEP}(G \setminus A) \leq \sum_s \text{NEP}_s(G \setminus A)$ where s goes through all the prime divisor of $|G/A|$. Let s be a prime and $x \in \text{EP}_s(G \setminus A)$. We have the following,*

- (1) $\text{NEP}_s(xA) \leq \text{NEP}_s(xA/U)(|W| - 1) \leq (s - 1)|A/F|e^2(|W| - 1)$.
- (2) $\text{NEP}_s(xA) \leq \text{NEP}_s(xA/U) \frac{|W|-1}{|W|^{1/s}-1} \leq (s - 1)|A/F|e^2 \frac{|W|-1}{|W|^{1/s}-1}$.
- (3) If $s = 2$, $e = p^k$ where p is an odd prime and $p \nmid |W|^{1/2} - 1$, then $\text{NEP}_2(xA) \leq \text{NEP}_2(xA/F)e^{\frac{|W|-1}{|W|^{1/2}-1}}$.
- (4) If $s = 2$, $A \lesssim \text{SL}(2, 3)$, then $\text{NEP}_2(xA) \leq 12$.
- (5) If $s = 2$, $A \lesssim S_3$, then $\text{NEP}_2(xA) \leq 3$.

Proof. \square

Lemma 2.14. *We count the number of elements of prime order in $\text{GL}(2, 2)$, $\text{SL}(2, 2)$, $\text{SL}(2, 3)$. We have the following facts.*

- (1) $\text{GL}(2, 2) \cong \text{SL}(2, 2) \cong S_3$, $\text{NEP}_2(S_3) = 3$ and $\text{NEP}_3(S_3) = 2$.
- (2) $|\text{SL}(2, 3)| = 24$, $\text{NEP}_2(\text{SL}(2, 3)) = 1$ and $\text{NEP}_3(\text{SL}(2, 3)) = 8$.

Proof. It is easy to check. \square

Lemma 2.15. *Suppose that H is a linear group on V_1 and S is a permutation group on Ω , with $|\Omega| = n$. $V = V_1 + V_2 + \cdots + V_n$ is an $H \wr S$ -module where V_i are copies of V_1 .*

- (1) *Let $V_1 = \mathbb{F}_2^2$, $H \cong S_3$, $n = 2$, $S \cong S_2$ and $G = S_3 \wr S_2$, then $|G| = 6^2 \cdot 2$, $\text{NEP}_2(G) = 21$, $\text{NEP}_3(G) = 8$, $\text{NPC}(G, 2, 3) = 6$, $\text{NPC}(G, 2, 2) = 15$, $\text{NPC}(G, 2, 1) = 0$, $\text{NPC}(G, 3, 2) = 4$.*
- (2) *Let $V_1 = \mathbb{F}_2^2$, $H \cong S_3$, $n = 3$, $S \cong S_3$ and $G = S_3 \wr S_3$, then $|G| = 6^4$, $\text{NEP}_2(G) = 135$, $\text{NEP}_3(G) = 98$, $\text{NPC}(G, 2, 5) = 9$, $\text{NPC}(G, 2, 4) = 45$, $\text{NPC}(G, 2, 3) = 81$, $\text{NPC}(G, 2, 2) = 0$, $\text{NPC}(G, 2, 1) = 0$.*
- (3) *Let $V_1 = \mathbb{F}_2^2$, $H \cong S_3$, $n = 4$, $S \cong S_4$ and $G = S_3 \wr S_4$, then $|G| = 6^4 \cdot 24$, $\text{NEP}(G) = 1883$, $\text{NPC}(G, 2, 7) = 12$, $\text{NPC}(G, 2, 6) = 90$, $\text{NPC}(G, 2, 5) = 324$, $\text{NPC}(G, 2, 4) = 513$, $\text{NPC}(G, 2, 3) = 0$, $\text{NPC}(G, 2, 2) = 0$, $\text{NPC}(G, 2, 1) = 0$.*
- (4) *Let $V_1 = \mathbb{F}_2^2$, $H \cong S_3$, $n = 5$, $S \cong F_{20}$ and $G = S_3 \wr F_{20}$, then $|G| = 6^5 \cdot 20$, $\text{NEP}(G) = 7169$, $\text{NPC}(G, 2, 8) = 90$, $\text{NPC}(G, 2, 6) = 585$, $\text{NPC}(G, 2, 4) = 0$, $\text{NPC}(G, 2, 2) = 0$.*
- (5) *Let $V_1 = \mathbb{F}_3^2$, $H \cong \text{SL}(2, 3)$, $n = 2$, $S \cong S_2$ and $G = \text{SL}(2, 3) \wr S_2$, then $|G| = 24^2 \cdot 2$, $\text{NEP}_2(G) = 27$, $\text{NEP}_3(G) = 80$, $\text{NPC}(G, 3, 3) = 16$.*

This follows from [6, Lemma 2.15].

Lemma 2.16. *Let n be an even integer and V be a standard symplectic vector space of dimension n of field \mathbb{F} . Let $G \in \text{SCRSp}(n, \mathbb{F})$.*

- (1) *Let $(n, \mathbb{F}) = (2, \mathbb{F}_2)$, then $G \lesssim S_3$, $|G| \leq 6$, $\text{NEP}_2(G) \leq 3$, $\text{NEP}_3(G) \leq 2$, $\text{NPC}(G, 2, 1) \leq 3$.*
- (2) *Let $(n, \mathbb{F}) = (4, \mathbb{F}_2)$, then $|G| \leq 6^2 \cdot 2$, $\text{NEP}_2(G) \leq 21$, $\text{NEP}_3(G) \leq 8$, $\text{NEP}_{\{2,3\}'}(G) \leq 4$, $\text{NEP}(G) \leq 29$, $\text{NPC}(G, 2, 3) \leq 6$, $\text{NPC}(G, 2, 2) \leq 15$, $\text{NPC}(G, 2, 1) = 0$, $\text{NPC}(G, 3, 2) \leq 4$.*
- (3) *Let $(n, \mathbb{F}) = (6, \mathbb{F}_2)$, then $|G| \leq 6^4$, $\text{NEP}_2(G) \leq 135$, $\text{NEP}_3(G) \leq 242$, $\text{NEP}_{\{2,3\}'}(G) \leq 6$, $\text{NPC}(G, 2, 5) \leq 9$, $\text{NPC}(G, 2, 4) \leq 45$, $\text{NPC}(G, 2, 3) \leq 108$, $\text{NPC}(G, 2, 2) = 0$, $\text{NPC}(G, 2, 1) = 0$.*
- (4) *Let $(n, \mathbb{F}) = (8, \mathbb{F}_2)$, then $|G| \leq 6^4 \cdot 24$, $\text{NEP}(G) \leq 1883$, $\text{NPC}(G, 2, 7) \leq 12$, $\text{NPC}(G, 2, 6) \leq 90$, $\text{NPC}(G, 2, 5) \leq 324$, $\text{NPC}(G, 2, 4) \leq 513$, $\text{NPC}(G, 2, 3) = 0$, $\text{NPC}(G, 2, 2) = 0$, $\text{NPC}(G, 2, 1) = 0$.*
- (5) *Let $(n, \mathbb{F}) = (2, \mathbb{F}_3)$, then $G \lesssim \text{SL}(2, 3)$ and $|G| \leq 24$. $\text{NEP}_2(G) \leq 1$ and $\text{NEP}_3(G) \leq 8$.*
- (6) *Let $(n, \mathbb{F}) = (4, \mathbb{F}_3)$, then $|G| \leq 24^2 \cdot 2$ and $\text{NEP}(G) \leq 107$, $\text{NEP}_2(G) \leq 95$, $\text{NEP}_3(G) \leq 95$, $\text{NEP}_5(G) \leq 64$ and G will have no elements with other prime order. Assume $G \not\lesssim \text{SL}(2, 3) \wr S_2$, then $\text{NPC}(G, 3, 3) = 0$.*

This follows from [6, Lemma 2.17].

3. MAIN THEOREM

Theorem 3.1. *Suppose that a finite solvable group G acts faithfully, irreducibly and quasi-primitively on a finite vector space V . By Theorem 2.2, G will have a uniquely determined normal subgroup E which is a direct product of extraspecial p -groups for various p and $e = \sqrt{|E/\mathbf{Z}(E)|}$.*

- (1) *Let $e = 16$. If $|W| \geq 7$, or if $|W| \geq 3$ and $b \geq 2$, then G will have at least one regular orbit on V .*

- (2) Let $e = 9$. If $|W| \geq 29$ and $|W|$ is a prime or when $|W| \geq 11^2$ and $|W|$ is a prime power, then G will have at least one regular orbit on V . If $|W| \geq 7$ and $b \geq 2$, then G will have at least one regular orbit on V .
- (3) Let $e = 8$. If $|W| \geq 19$ and $|W|$ is a prime or when $|W| \geq 7^2$ and $|W|$ is a prime power, then G will have at least one regular orbit on V . If $|W| \geq 5$ and $b \geq 2$, then G will have at least one regular orbit on V .
- (4) Let $e = 4$. If $|W| \geq 71$ and $|W|$ is a prime or when $|W| \geq 11^2$ and $|W|$ is a prime power, then G will have at least one regular orbit on V . If $b \geq 2$, and $|W| \geq 11$ is a single prime or $|W| \geq 5^2$ is a prime power, then G will have at least one regular orbit on V .
- (5) Let $e = 3$. If $|W| \geq 43$ and $|W|$ is a prime or when $|W| = 11^2$ or when $|W| \geq 16^2$ and $|W|$ is a prime power, then G will have at least one regular orbit on V . If $|W| \geq 13$ and $b \geq 2$, then G will have at least one regular orbit on V .
- (6) Let $e = 2$. If $|W| \geq 23$ and $|W|$ is a prime or when $|W| \geq 17^2$ and $|W|$ is a prime power, then G will have at least one regular orbit on V . If $|W| \geq 9$ and $b \geq 2$, then G will have at least one regular orbit on V .

Proof. In order to show that G has at least one regular orbit on V it suffices to check that

$$\left| \bigcup_{P \in \text{SP}(G)} \mathbf{C}_V(P) \right| < |V|.$$

In the following cases we will divide the set $\text{SP}(G)$ into a union of sets A_i . Clearly

$$\left| \bigcup_{P \in \text{SP}(G)} \mathbf{C}_V(P) \right| \leq \sum_i \left| \bigcup_{P \in A_i} \mathbf{C}_V(P) \right|.$$

We will find $\beta_i < e$ such that $|\mathbf{C}_V(P)| \leq |W|^{\beta_i b}$ for all $P \in A_i$. We will find a_i such that $|A_i| \leq a_i$. Also we will find B such that $|G| \leq B$. Since $|V| = |W|^{eb}$ it suffices to check that

$$\sum_i a_i \cdot (|W|^{\beta_i b} - 1) < |W|^{eb} - 1.$$

We call this inequality \star .

Let $e = 16$. Thus $2 \mid |W| - 1$ and $A/F \in \text{SCRS}_{\text{p}}(8, 2)$. $|A/F| \leq 6^4 \cdot 24$ by Lemma 2.16(4).

Define $A_1 = \{\langle x \rangle \mid x \in \text{EP}_2(F \setminus U)\}$. Thus for all $P \in A_1$, $|\mathbf{C}_V(P)| \leq |W|^{8b}$ by Lemma 2.4(1) and we set $\beta_1 = 8$. Since $F = E \vee U$ and $U \subseteq \mathbf{Z}(F)$, $|A_1| \leq 2^8 = a_1$.

Define $A_2 = \{\langle x \rangle \mid x \in \text{EP}_2(A \setminus F), x \text{ is a good element and } \mathbf{C}_{\bar{E}}(x) = 2^6\}$. Let $\langle x \rangle \in A_2$. Since x is a good element, then $|\mathbf{C}_V(x)| \leq |W|^{\lfloor \frac{1}{2}(16+8) \rfloor b} = |W|^{12b}$ by Lemma 2.6(2). Thus for all $P \in A_2$, $|\mathbf{C}_V(P)| \leq |W|^{12b}$ and we set $\beta_2 = 12$. $|A_2| \leq 90 \cdot 2^2 \cdot 2 = a_2$ by Lemma 2.16(4) and Lemma 2.13(4).

Define $A_3 = \{\langle x \rangle \mid x \in \text{EP}_2(A \setminus F), x \text{ is a good element and } \mathbf{C}_{\bar{E}}(x) = 2^4\}$. Let $\langle x \rangle \in A_3$. Since x is a good element, $|\mathbf{C}_V(x)| \leq |W|^{\lfloor \frac{1}{2}(16+4) \rfloor b} = |W|^{10b}$ by Lemma 2.6(2). Thus for all $P \in A_3$, $|\mathbf{C}_V(P)| \leq |W|^{10b}$ and we set $\beta_3 = 10$. $|A_3| \leq 513 \cdot 2^4 \cdot 2 = a_3$ by Lemma 2.16(4) and Lemma 2.13(4).

Define $A_4 = \{\langle x \rangle \mid x \in \text{EP}_2(A \setminus F) \text{ and } x \text{ is bad; or } x \in \text{EP}_s(A \setminus F) \text{ for all primes } s \geq 3\}$. Let $\langle x \rangle \in A_4$. If $x \in \text{EP}_2(A \setminus F)$ and x is bad, then $|\mathbf{C}_V(x)| \leq |W|^{8b}$ by Lemma 2.6(1)

and $\text{NEP}_2(xF) \leq 2^8 \cdot 2$ by Lemma 2.13(3). If $x \in \text{EP}_s(A \setminus F)$ for a prime $s \geq 3$, then $|\mathbf{C}_V(x)| \leq |W|^{8b}$ by Lemma 2.4(3) and $\text{NEP}_s(xF) \leq 2^8 \cdot s$ by Lemma 2.13(3). Thus for all $P \in A_4$, $|\mathbf{C}_V(P)| \leq |W|^{8b}$ and we set $\beta_4 = 8$. $|A_4| \leq 1883 \cdot 2^8 \cdot 2 = a_4$ by Lemma 2.16(4).

Define $A_5 = \{\langle x \rangle \mid x \in \text{EP}(G \setminus A)\}$. Thus for all $P \in A_5$, $|\mathbf{C}_V(P)| \leq |W|^{8b}$ by Lemma 2.4(5) and we set $\beta_5 = 8$. We set $a_5 = \text{Div}(\dim(W)) \cdot 6^4 \cdot 24 \cdot 2^8 \cdot (|W|^{1/2} + 1)$ by Lemma 2.10.

It is routine to check that \star is satisfied when $|W| \geq 7$. \star is also satisfied if $|W| \geq 3$ and $b \geq 2$.

Let $e = 9$. Thus $3 \mid |W| - 1$ and $A/F \in \text{SCRSp}(4, 3)$. $|A/F| \leq 6^4 \cdot 24$ by Lemma 2.16(4).

Define $A_1 = \{\langle x \rangle \mid x \in \text{EP}_3(F \setminus U)\}$. Thus for all $P \in A_1$, $|\mathbf{C}_V(P)| \leq |W|^{3b}$ by Lemma 2.4(1) and we set $\beta_1 = 3$. $|A_1| \leq 3^5/2 = a_1$.

Define $A_2 = \{\langle x \rangle \mid x \in \text{EP}_2(A \setminus F)\}$. Thus for all $P \in A_2$, $|\mathbf{C}_V(P)| \leq |W|^{6b}$ by Lemma 2.4(2) and we set $\beta_2 = 6$. $|A_2| \leq 95 \cdot 3^4 \cdot 2 = a_2$ (this could be improved by studying the 2 elements in A/F) by Lemma 2.16(6) and Lemma 2.13(2).

Define $A_3 = \{\langle x \rangle \mid x \in \text{EP}_3(A \setminus F) \text{ and } |\mathbf{C}_V(x)| \geq |W|^{6b}\}$. Let $x \in \text{EP}_3(A \setminus F)$. If x is a bad element, then $|\mathbf{C}_V(x)| \leq |W|^{3b}$ by Lemma 2.6(1). If x is a good element and $|\mathbf{C}_{\bar{E}}(x)| \leq 3^2$, then $|\mathbf{C}_V(x)| \leq |W|^{\lfloor \frac{1}{3}(9+2 \cdot 3) \rfloor b} = |W|^{5b}$ by Lemma 2.6(2). Thus for all $1 \neq x \in P \in A_3$, x is a good element, $|\mathbf{C}_{\bar{E}}(x)| = |\chi(x)|^2 = 3^3$ and $|W|^{6b} \leq |\mathbf{C}_V(x)| = |W|^{\lfloor \frac{1}{3}(9+\chi(x)+\overline{\chi(x)}) \rfloor b} \leq |W|^{\lfloor \frac{1}{3}(9+2 \cdot 3 \cdot \sqrt{3}) \rfloor b} = |W|^{6b}$ by Lemma 2.6(2) and Lemma 2.6(3) and we set $\beta_3 = 6$. Since $|\chi(x)| = 3 \cdot \sqrt{3}$ and $\chi(x) + \overline{\chi(x)} = 9$, we know that $\chi(x) = \frac{9}{2} \pm \frac{3\sqrt{3}}{2}i$. We may choose an element $z \in U$, $o(z) = 3$ such that $\chi(xz) = -\frac{9}{2} \pm \frac{3\sqrt{3}}{2}i$ and $\chi(xz^2) = \mp 3\sqrt{3}i$. Now we have $|\mathbf{C}_V(xz)| \leq 1$, $|\mathbf{C}_V(xz^2)| \leq |W|^3$ and thus $|A_3| \leq \text{NPC}(A/F, \bar{E}, 3, 3) \cdot |F/U|/2 \leq 16 \cdot 3^4/2 = a_3$ by Lemma 2.16(6).

Define $A_4 = \{\langle x \rangle \mid x \in \text{EP}_3(A \setminus F) \text{ and } |\mathbf{C}_V(x)| \leq |W|^5\}$. We set $\beta_4 = 6$. Also $|A_4| \leq 95 \cdot 3^5/2 = a_4$ by Lemma 2.16(6) and Lemma 2.13(3).

Define $A_5 = \{\langle x \rangle \mid x \in \text{EP}_s(H \cap (G \setminus A))\}$. Thus for all $P \in A_5$, $|\mathbf{C}_V(P)| \leq |W|^{4.5b}$ by Lemma 2.4(4) and we set $\beta_5 = 4.5$. $|A_5| \leq \text{Div}(\dim(W)) \cdot 2 \cdot 24^2 \cdot 3^4 \cdot (|W| - 1) = a_5$ by Lemma 2.16(6). For 2 we have $|A_5| \leq \text{Div}(\dim(W)) \cdot 2 \cdot 24^2 \cdot 3^4 \cdot (|W|^{1/2} + 1) = a_5$ by Lemma 2.16(6), Lemma 2.10 and Lemma 2.8.

It is routine to check that \star is satisfied when $|W| \geq 29$ and $|W|$ is a prime or when $|W| \geq 121$ and $|W|$ is a prime power. Also \star is satisfied if $|W| \geq 7$ and $b \geq 2$.

Let $e = 8$. Thus $2 \mid |W| - 1$ and $A/F \in \text{SCRSp}(6, 2)$. $|A/F| \leq 6^4$ by Lemma 2.16(3).

Define $A_1 = \{\langle x \rangle \mid x \in \text{EP}_2(F \setminus U)\}$. Thus for all $P \in A_1$, $|\mathbf{C}_V(P)| \leq |W|^{4b}$ by Lemma 2.4(1) and we set $\beta_1 = 4$. Since $F = E \vee U$ and $U \subseteq \mathbf{Z}(F)$, $|A_1| \leq 2^8 = a_1$.

Define $A_2 = \{\langle x \rangle \mid x \in \text{EP}_2(A \setminus F), x \text{ is a good element and } \mathbf{C}_{\bar{E}}(x) = 2^4\}$. Let $\langle x \rangle \in A_2$. Since x is a good element, $|\mathbf{C}_V(x)| \leq |W|^{\lfloor \frac{1}{2}(8+4) \rfloor b} = |W|^{6b}$ by Lemma 2.6(2). Thus for all $P \in A_2$, $|\mathbf{C}_V(P)| \leq |W|^{6b}$. $|A_2| \leq 45 \cdot 2^2 \cdot 2$ by Lemma 2.16(3) and Lemma 2.13(4). Since $\mathbf{C}_{\bar{E}}(x) = |\chi(x)|^2 = 2^4$, we know that $\chi(x) = \pm 4$. If $\chi(x) = 4$, then for $z \in U$, $o(z) = 2$, $\chi(xz) = -4$, and we have $|\mathbf{C}_V(xz)| \leq |W|^2$. Thus A_2 can be split into two sets A_{21} and A_{22} . Where if $\langle x \rangle \in A_{21}$, then $|\mathbf{C}_V(x)| \leq |W|^{6b}$ and we set $\beta_{21} = 6$. If $\langle x \rangle \in A_{22}$, then $|\mathbf{C}_V(x)| \leq |W|^{2b}$ and we set $\beta_{22} = 2$. Also $|A_{21}| = |A_{22}| \leq 45 \cdot 2^2$ and we set $a_{21} = a_{22} = 45 \cdot 2^2$.

Define $A_3 = \{\langle x \rangle \mid x \in \text{EP}_2(A \setminus F) \text{ and } x \text{ is bad}\}$. Let $\langle x \rangle \in A_3$, then $|\mathbf{C}_V(x)| \leq |W|^{4b}$ by Lemma 2.6(1). Thus for all $P \in A_3$, $|\mathbf{C}_V(P)| \leq |W|^{4b}$ and we set $\beta_3 = 4$. $|A_3| \leq 135 \cdot 2^6 \cdot 2 = a_3$ by Lemma 2.16(3) and Lemma 2.13(3). need more argument

Define $A_4 = \{\langle x \rangle \mid x \in \text{EP}_s(A \setminus F) \text{ for all primes } s \geq 3\}$. Let $\langle x \rangle \in A_4$, then $|\mathbf{C}_V(x)| \leq |W|^{4b}$ by Lemma 2.4(3). Thus for all $P \in A_4$, $|\mathbf{C}_V(P)| \leq |W|^{4b}$ and we set $\beta_4 = 4$. $|A_4| \leq 248 \cdot 2^6 \cdot 3/2 = a_4$ by Lemma 2.16(3) and Lemma 2.13(3).

Define $A_5 = \{\langle x \rangle \mid x \in \text{EP}(G \setminus A)\}$. Thus for all $P \in A_5$, $|\mathbf{C}_V(P)| \leq |W|^{4b}$ by Lemma 2.4(5) and we set $\beta_5 = 4$. We set $a_5 = \text{Div}(\dim(W)) \cdot 6^4 \cdot 2^6 \cdot (|W|^{1/2} + 1)$ by Lemma 2.10.

It is routine to check that \star is satisfied when $|W| \geq 19$ and $|W|$ is a prime or when $|W| \geq 49$ and $|W|$ is a prime power. Also \star is satisfied if $|W| \geq 5$ and $b \geq 2$.

Let $e = 4$. Thus $2 \mid |W| - 1$ and $A/F \in \text{SCRS}\text{p}(4, 2)$ and $|A/F| \leq 6^2 \cdot 2$ by Lemma 2.16(2).

Define $A_1 = \{\langle x \rangle \mid x \in \text{EP}_2(F \setminus U)\}$. Thus for all $P \in A_1$, $|\mathbf{C}_V(P)| \leq |W|^{2b}$ by Lemma 2.4(1) and we set $\beta_1 = 2$. Since $F = E \vee U$ and $U \subseteq \mathbf{Z}(F)$, $|A_1| \leq 2^5 = a_1$.

Define $A_2 = \{\langle x \rangle \mid x \in \text{EP}_2(A \setminus F), x \text{ is a good element and } \mathbf{C}_{\bar{E}}(x) = 2^2\}$. Let $\langle x \rangle \in A_2$. Since x is a good element, $|\mathbf{C}_V(x)| \leq |W|^{\lfloor \frac{1}{2}(4+2) \rfloor b} = |W|^{3b}$ by Lemma 2.6(2). Thus for all $P \in A_2$, $|\mathbf{C}_V(P)| \leq |W|^{3b}$. $|A_2| \leq 15 \cdot 2^2 \cdot 2$ by Lemma 2.16(2) and Lemma 2.13(4). Since $\mathbf{C}_{\bar{E}}(x) = |\chi(x)|^2 = 2^2$, we know that $\chi(x) = \pm 2$. If $\chi(x) = 2$, then for $z \in U$, $o(z) = 2$, $\chi(xz) = -2$, and we have $|\mathbf{C}_V(xz)| \leq |W|$. Thus A_2 can be split into two sets A_{21} and A_{22} . Where if $\langle x \rangle \in A_{21}$, then $|\mathbf{C}_V(x)| \leq |W|^{3b}$ and we set $\beta_{21} = 3$. If $\langle x \rangle \in A_{22}$, then $|\mathbf{C}_V(x)| \leq |W|^b$ and we set $\beta_{22} = 3$. Also $|A_{21}| = |A_{22}| \leq 15 \cdot 2^2$ and we set $a_{21} = a_{22} = 15 \cdot 2^2$.

Define $A_3 = \{\langle x \rangle \mid x \in \text{EP}_2(A \setminus F) \text{ and } x \text{ is bad}\}$. Let $\langle x \rangle \in A_3$. Since x is a bad element, $|\mathbf{C}_V(x)| \leq |W|^{2b}$ by Lemma 2.6(1). Thus for all $P \in A_2$, $|\mathbf{C}_V(P)| \leq |W|^{2b}$ and we set $\beta_3 = 2$. $|A_3| \leq 21 \cdot 2^4 \cdot 2 = a_3$ by Lemma 2.16(2) and Lemma 2.13(3).

Define $A_4 = \{\langle x \rangle \mid x \in \text{EP}_3(A \setminus F)\}$. Let $\langle x \rangle \in A_4$, then $|\mathbf{C}_V(x)| \leq |W|^{2b}$ by Lemma 2.4(3). Thus for all $P \in A_4$, $|\mathbf{C}_V(P)| \leq |W|^{2b}$ and we set $\beta_4 = 2$. $|A_4| \leq (4 \cdot 2^2 + 4 \cdot 2^4) \cdot D_3(|W| - 1)/2 = a_4$ by Lemma 2.16(2) and Lemma 2.13(3).

Define $A_5 = \{\langle x \rangle \mid x \in \text{EP}(G \setminus A)\}$. Thus for all $P \in A_5$, $|\mathbf{C}_V(P)| \leq |W|^{2b}$ by Lemma 2.4(5) and we set $\beta_5 = 2$. We set $a_5 = \text{Div}(\dim(W)) \cdot 2 \cdot 6^2 \cdot 2^4 \cdot (|W|^{1/2} + 1)$ by Lemma 2.10.

It is routine to check that \star is satisfied when $|W| \geq 71$ and $|W|$ is a prime or $|W| \geq 11^2$ and $|W|$ is a prime power. Also \star is satisfied if $b \geq 2$, $|W| \geq 11$ and $|W|$ is a prime or $|W| \geq 5^2$ and $|W|$ is a prime power.

Let $e = 3$. Thus $2 \mid |W| - 1$ and $A/F \lesssim \text{SL}(2, 3)$ and $|A/F| \leq 24$ by Lemma 2.16(5).

Define $A_1 = \{\langle x \rangle \mid x \in \text{EP}_3(F \setminus U)\}$. Thus for all $P \in A_1$, $|\mathbf{C}_V(P)| \leq |W|^b$ by Lemma 2.4(1) and we set $\beta_1 = 1$. Since $F = E \vee U$ and $U \subseteq \mathbf{Z}(F)$, $|A_1| \leq 3^3/2 = a_1$.

Define $A_2 = \{\langle x \rangle \mid x \in \text{EP}_2(A \setminus F)\}$. Let $\langle x \rangle \in \text{EP}_2(A \setminus F)$, then $|\mathbf{C}_V(x)| \leq |W|^{2b}$ by Lemma 2.4(4). Thus for all $P \in A_2$, $|\mathbf{C}_V(P)| \leq |W|^{2b}$ and we set $\beta_2 = 2$. $|A_2| \leq 1 \cdot 3^2 \cdot D_2(|W| - 1) = a_2$ by Lemma 2.16(5) and Lemma 2.13(3).

Define $A_3 = \{\langle x \rangle \mid x \in \text{EP}_3(A \setminus F)\}$. Let $\langle x \rangle \in \text{EP}_3(A \setminus F)$. If x is a bad element, then $|\mathbf{C}_V(x)| \leq |W|^b$ by Lemma 2.6(1). Let $z \in U$ and $o(z) = 3$, by the definition in 2.6, xz and xz^2 are all bad elements, $|\mathbf{C}_V(xz)| \leq |W|^b$ and $|\mathbf{C}_V(xz^2)| \leq |W|^b$.

If x is a good element, by the definition in 2.6, xz and xz^2 are all good elements. Since $|\mathbf{C}_{\bar{E}}(x)| = 3$, $|\mathbf{C}_V(x)| \leq |W|^{\lfloor \frac{1}{3}(3+2\sqrt{3}) \rfloor} = |W|^{2b}$ by Lemma 2.6(2). If $|\mathbf{C}_V(x)| = |W|^{2b}$, then $|\mathbf{C}_{\bar{E}}(x)| = |\chi(x)|^2 = 3$ and $|W|^{2b} = |\mathbf{C}_V(x)| = |W|^{\lfloor \frac{1}{3}(3+\chi(x)+\overline{\chi(x)}) \rfloor b} \leq |W|^{\lfloor \frac{1}{3}(3+2\sqrt{3}) \rfloor b} = |W|^{2b}$

by Lemma 2.6(2) and Lemma 2.6(3). Since $|\chi(x)| = \sqrt{3}$ and $\chi(x) + \overline{\chi(x)} = 3$, we know that $\chi(x) = \frac{3}{2} \pm \frac{\sqrt{3}}{2}i$. We may choose an element $z \in U$, $o(z) = 3$ such that $\chi(xz) = -\frac{3}{2} \pm \frac{\sqrt{3}}{2}i$ and $\chi(xz^2) = \mp\sqrt{3}i$. And we have $|\mathbf{C}_V(xz)| \leq 1$, $|\mathbf{C}_V(xz^2)| \leq |W|^b$.

Based on the previous discussion, we can see that the worst situation is that all the elements in A_3 are good, and in this case A_3 can be split into three sets A_{31} , A_{32} and A_{33} . Where if $\langle x \rangle \in A_{31}$, then $|\mathbf{C}_V(x)| \leq |W|^{2b}$ and we set $\beta_{31} = 2$. If $\langle x \rangle \in A_{32}$, then $|\mathbf{C}_V(x)| \leq |W|^b$ and we set $\beta_{32} = 1$. If $\langle x \rangle \in A_{33}$, then $|\mathbf{C}_V(x)| \leq 1$ and we set $\beta_{33} = 0$. Also $|A_{31}| = |A_{32}| = |A_{33}| \leq 8 \cdot 3^2/2$ and we set $a_{31} = a_{32} = a_{33} = 8 \cdot 3^2/2$ by Lemma 2.16(5) and Lemma 2.13(3).

Define $A_5 = \{\langle x \rangle \mid x \in \text{EP}(G \setminus A)\}$. Thus for all $P \in A_5$, $|\mathbf{C}_V(P)| \leq |W|^{1.5b}$ by Lemma 2.4(5) and we set $\beta_5 = 1.5$. We set $a_4 = \text{Div}(\dim(W)) \cdot 24 \cdot 3^2 \cdot (|W| - 1)$. If 2 , $a_5 = 0.5 \cdot 24 \cdot 3^2 \cdot (|W|^{1/2} + 1)$.

It is routine to check that \star is satisfied when $|W| \geq 61$. $|W| \neq 11^2$, $|W| \geq 16^2$ and $|W|$ is a prime power. Also \star is satisfied if $|W| \geq 13$ and $b \geq 2$.

Let $|W| = 43$, the estimate of $|A_3|$ can be improved as the following. If A_3 is empty, then $a_3 = 0$. Otherwise, let $\langle x \rangle \in A_3$, then $x\langle E \rangle$ is a subgroup of A . Since $3^2 \nmid |W| - 1$, for every $1 \neq y \in \text{EP}_3(A \setminus F)$, $y \in A/F \cdot E \setminus E$. Using GAP, one checks that the number of elements of order 3 in $A/F \cdot E \setminus E$ is at most 144. $144/3 = 48$ is better than $8 \cdot 3^2 = 72$ for a_3 . The will take care of this case. Thus the next one is $|W| = 37$.

Let $e = 2$. Thus $2 \mid |W| - 1$, $A/F \lesssim S_3$ and $|A/F| \leq 6$ by Lemma 2.16(1).

Define $A_1 = \{\langle x \rangle \mid x \in \text{EP}_2(F \setminus U)\}$. Thus for all $P \in A_1$, $|\mathbf{C}_V(P)| \leq |W|^b$ by Lemma 2.4(1) and we set $\beta_1 = 1$. Since $F = E \vee U$ and $U \subseteq \mathbf{Z}(F)$, $|A_1| \leq 4 = a_1$.

Define $A_2 = \{\langle x \rangle \mid x \in \text{EP}_2(A \setminus F)\}$. Let $\langle x \rangle \in A_2$. Since $\mathbf{C}_{\bar{E}}(x) = 2$, x is a bad element and $|\mathbf{C}_V(x)| \leq |W|^b$ by Lemma 2.6(1). Thus for all $P \in A_2$, $|\mathbf{C}_V(P)| \leq |W|^b$ and we set $\beta_2 = 1$. $|A_2| \leq 3 \cdot 2^2 \cdot 2 = a_2$ by Lemma 2.16(1) and Lemma 2.13(3).

Define $A_3 = \{\langle x \rangle \mid x \in \text{EP}_3(A \setminus F)\}$. Let $\langle x \rangle \in A_3$, then $|\mathbf{C}_V(x)| \leq |W|^b$ by Lemma 2.4(3). Thus for all $P \in A_3$, $|\mathbf{C}_V(P)| \leq |W|^b$ and we set $\beta_3 = 1$. $|A_3| \leq 2 \cdot 2^2 \cdot D_3(|W| - 1)/2 = a_3$ by Lemma 2.16(1) and Lemma 2.13(3).

Define $A_4 = \{\langle x \rangle \mid x \in \text{EP}(G \setminus A)\}$. Thus for all $P \in A_4$, $|\mathbf{C}_V(P)| \leq |W|^b$ by Lemma 2.4(5) and we set $\beta_5 = 1$. We set $a_4 = \text{Div}(\dim(W)) \cdot 6 \cdot 2^2 \cdot (|W|^{1/2} + 1)$ by Lemma 2.10. For if $o(x) = 2$, one may reduce the estimate to $4 \cdot 2^2 \cdot (|W|^{1/2} + 1)$ by studying S_3 , and reduce further to $3 \cdot 2^2 \cdot (|W|^{1/2} + 1)$ by studying $Q_8 \rtimes Z_4$.

It is routine to check that \star is satisfied when $|W| \geq 41$ and $|W|$ is a prime, or when $|W| \geq 17^2$ and $|W|$ is a prime power. Also if $b \geq 2$, \star is satisfied when $|W| \geq 7$ and $|W|$ is a prime, or when $|W| \geq 9$ and $|W|$ is a prime power.

And by using GAP, one may get rid of the cases when $|W| = 37, 31, 29, 23$ and $b = 1$. \square

Remak: The proof of the main result is written in a way that unified decompositions and arguments work for many different cases. In some cases, it is possible to refine the arguments to improve the bound on $|V|$ a little bit.

4. ACKNOWLEDGEMENT

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